

**DIRAC EQUATIONS  
LIGHT CONE SUPERSYMMETRY  
and  
SUPERCONFORMAL ALGEBRAS**

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After a brief historical survey that emphasizes the role of the algebra obeyed by the Dirac operator, we examine an algebraic Dirac operator associated with Lie algebras and Lie algebra cosets. For symmetric cosets, its “massless” solutions display non-relativistic supersymmetry, and can be identified with the massless degrees of freedom of some supersymmetric theories:  $N = 1$  supergravity in eleven dimensions (M-theory), type IIB string theory in ten and four dimensions, and in four dimensions,  $N = 8$  supergravity,  $N = 4$  super-Yang-Mills, and the  $N = 1$  Wess-Zumino multiplet. By generalizing this Dirac operator to the affine case, we generate superconformal algebras associated with cosets  $\mathfrak{g}/\mathfrak{h}$ , where  $\mathfrak{h}$  contains the *space* little group. Only for eleven dimensional supergravity is  $\mathfrak{h}$  simple. This suggests, albeit in a non-relativistic setting, that these may be the limit of theories with underlying two-dimensional superconformal structure.

## 1 Historical Remarks

The development of supersymmetry in the Western world and in the Eastern world of the time was very different. In this book there are a number of contributions that tell the story from the Eastern side. In the Soviet scientific society the scientists had one freedom that scientists in the West lacked and still lack (perhaps the only real freedom that Eastern scientists had), and that was to spend time also on esoteric questions. They did not have to be scrutinized by funding agencies every now and then. As one consequence of this fact, the Soviet scientists of the time were often very well trained in basic formalisms. They had in many cases a superior knowledge in fermion fields and their treatment in quantum field theory, which is very well witnessed in the brilliant book by Berezin <sup>1</sup>. This background was fertile ground for work leading up to

supersymmetry. To us it seems that the development resulting in the work of Yurij Golfand and Evgenij Likhtman <sup>2</sup> was a natural one (without meaning to underestimate its fundamental importance). By studying quantum field theory per se without the need to directly correlate it to particle physics of the time it is a clever but also a natural idea to look for a symmetry between bosons and fermions. Quantum field theory that was essentially abandoned in the West because of its severe difficulties to go beyond QED, certainly needed new elements. To match the number of bosonic degrees of freedom to the fermionic ones was not a natural idea in the West in the 60's which was in the midst of the Quark Model and Current Algebra. Of course, a superalgebra in terms of the anticommutator of the two Dirac operators was certainly known but not given much importance. Abdus Salam and collaborators <sup>3</sup> had been interested in functional integrations over spinors in the 50's but had not completed the issue as Berezin had done. However, once four-dimensional supersymmetry became popular in the West, Abdus Salam became very active and had a huge experience to draw from. However, Yurij Golfand might very well have been the only one who seriously pushed a four-dimensional fermionic symmetry in quantum field theory in the late 60's.

In the West the abandonment of quantum field theory for the strong interactions led to S-matrix theory and to a detailed study of the possible analytic structure of this matrix. More and more resonances were found in the strong interactions experiments, and in 1968 Gabriele Veneziano <sup>4</sup> managed to construct an S-matrix (a dual resonance model) for four bosons that incorporated infinitely many resonances with the necessary analytic structure for a Born-term. This was the starting point for string theory in that slightly later it was understood to describe scattering of string states. The important point we want to make is that this development was very much influenced by the experimental discoveries of the time. Particle physics was still quite young and the theorists in the West were very much involved in constructing models for the experimental data that kept pouring out. After Veneziano's fundamental work it was so natural to ask how to implement fermions into dual resonance model. This was not an easy task since the model was formulated just as a scattering amplitude. The key to solve this was the realization of the underlying two-dimensional world sheet. Veneziano's model described scattering based on bosonic degrees of freedom on the world-sheet. To introduce space-time spinors the natural thing was then to introduce fermions on the world sheet <sup>5</sup>. In this process it was realized that two-dimensional supersymmetry was needed to get a viable model and this was the start of supersymmetry in the West. The symmetry even had to be a superconformal symmetry and the ordinary supersymmetry came out as a zero-mode. Supersymmetry hence was

a result of the direct strive to find a model to describe the strong interactions. The fact that the two first papers of supersymmetry in the West and the East were submitted at about the same time in late 1970 is quite remarkable.

In the 30 years since this development took place the two lines of theoretical research described above have converged. The development of supersymmetric gauge field theories and superstring theory took us out of contact with much of the experimental data of today. The success of superstring theory builds to a large extent on our increased mathematical insight and this has spurred us to look for new structures. The science of today is very much depending on our attempts to find those and now we cannot only rely on experimental data to lead our way. The studies of superstring theory and supersymmetric gauge theory per se have led us to all the wonderful discoveries of recent years. It is hence very much worthwhile to study new structures in mathematics and try to connect such developments to modern string theory. The understanding of M-theory as an 11-dimensional theory which is not a regular string theory has made us go back to a study of group theory to look for alternatives to supersymmetry and in this contribution we will discuss some recent studies of ours which we hope will shed new light on even more fundamental levels of physics than superstrings.

## 2 Supersymmetry and the Dirac Equation

It was known since the early days of quantum mechanics that the Dirac equation

$$\not{p}\Psi = 0 , \tag{1}$$

contains a new type of anticommuting symmetry, since the Dirac operator obeys the superalgebra

$$\{ \not{p}\not{p} \} = p^2 , \quad [ \not{p} p^2 ] = [ p^2, p^2 ] = 0 . \tag{2}$$

However, in ordinary quantum field theory this symmetry was not really used. In the development of the Veneziano Model in the late 60's it was however realized that the modes of  $p^2$  when taken to be a function of a complex variable (of the Euclideanized world-sheet) span the two-dimensional conformal algebra, the Virasoro algebra <sup>6</sup>. This algebra is the underlying symmetry algebra of the model and it was natural to seek to generalize this algebra to also contain fermions <sup>5</sup>. The key then was to generalize the  $\gamma$ -matrices to also be functions of this complex variable. This led to the superconformal algebra and eventually to the superstring model. As a by-product one got the 2-dimensional

supersymmetry algebra. Strings naturally demanded 2-dimensional supersymmetry and eventually it was also realized that they demand supersymmetry in the target space.

In the struggle to find realistic string models the first attempts were to generalize the superconformal algebra. Such a generalization was finally found in the  $N = 2$  string <sup>7</sup> where  $N$  refers to the underlying superconformal algebra. When the superstring was properly understood as unifying theories of all interactions it was clear that the original one was the most promising and the challenge was then to understand the 10-dimensional space on which it lives. In this way we obtained the five superstring models and opened up the road to compactifications. With the advent of M-theory as a non-standard string theory we have found it worthwhile to study again the roots of supersymmetry and its role in string theory. The superconformal generator is part of the superconformal algebra used as the gauge algebra of the model and since in string theory the target manifold contains the four-dimensional Minkowski space the zero mode part gives that states in the R-sector must satisfy the Dirac equation in Minkowski space. In the sequel we will study Dirac equations on other types of manifolds and check when the “massless” states coincide with interesting multiplets.

### 3 Kostant’s Operator

As we have seen, in formulating supersymmetric theories, it is natural to focus on the Dirac operator. It is then interesting to note that for any manifold with dimension equal to that of a Lie algebra, or for any coset generated Lie algebra embeddings, there exists a natural Dirac-like operator <sup>8</sup>. For compact Lie algebras, this Dirac operator has only “massive” solutions. However, the Dirac operator associated with symmetric cosets has “massless” solutions, some of which display manifest supersymmetry. In the sequel we will use this method to generate supermultiplets. The method will be group theoretical and the relation to space-time symmetries will not be obvious.

#### 3.1 Lie Algebras

Consider a  $D$  dimensional Lie algebra,  $\mathfrak{g}$ , defined by the commutation relations

$$[T^a, T^b] = if^{abc}T^c, \quad (3)$$

with  $a, b, c = \dots, D$ . On this  $D$ -dimensional vector space, the  $D$  Dirac matrices which satisfy the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}, \quad (4)$$

are realized in terms of  $2^D \times 2^D$  matrices (for  $D$  even). We now define the Kostant-Dirac operator

$$\mathcal{K} \equiv \sum_{a=1}^D \gamma^a T^a - \frac{i}{2 \cdot 3!} \sum_{a,b,c}^D \gamma^{abc} f^{abc} , \quad (5)$$

where  $\gamma^{abc} = \gamma^{[a} \gamma^b \gamma^{c]}$  is totally antisymmetrized. Because of the term cubic in the Clifford algebra, it has the remarkable property that

$$\{\mathcal{K}, \mathcal{K}\} = 2C_2 + \frac{D}{12} C_2^{\text{adjt}} , \quad (6)$$

where  $C_2$  is the quadratic Casimir operator of the representation of the  $T^a$  matrices, and  $C_2^{\text{adjt}}$  is the quadratic Casimir of the adjoint representation of the algebra. Because the algebra is compact, the square is the sum of two positive terms, and the solution to the “massless” equation  $\mathcal{K} \Psi = 0$  is trivial, although there are many states on which  $\mathcal{K}$  takes on a constant value. We note that

$$[L^a, \mathcal{K}] = 0 , \quad (7)$$

where the diagonal matrix operators

$$L^a = T^a + S^a , \quad (8)$$

with

$$S^a = -\frac{i}{4} \gamma^{bc} f^{abc} , \quad (9)$$

generate the representation of the algebra  $\mathfrak{g}$  on the spinor space of  $SO(D)$ . Thus the solutions of  $(\mathcal{K} - \mu)\Psi = 0$  will assemble in full multiplets of  $\mathfrak{g}$  for special values of the “mass”  $\mu$ .

### 3.2 Lie Algebra Cosets

Consider any Lie algebra,  $\mathfrak{g}$ , generated by the  $T^a$ ,  $a = 1, 2, \dots, D$ . Inside that Lie algebra lies a subalgebra,  $\mathfrak{h} \subset \mathfrak{g}$ , generated by  $T^i$ ,  $i = 1, 2, \dots, d$ . We label the remaining  $(D - d)$  generators as  $T^m$ . The commutation relations are

$$\begin{aligned} [T^i, T^j] &= if^{ijk} T^k , \\ [T^i, T^m] &= if^{imn} T^n , \\ [T^m, T^n] &= if^{mnj} T^j + if^{mnp} T^p . \end{aligned} \quad (10)$$

Consider the  $(D - d)$  even-dimensional coset space, where the  $\gamma^m$  are  $2^{D-d} \times 2^{D-d}$  matrices that satisfy

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn} . \quad (11)$$

The Kostant operator associated with the coset is defined as

$$\mathcal{K} \equiv \sum_{m=1}^{D-d} \gamma^m T^m - \frac{i}{2 \cdot 3!} \sum_{m,n,p}^{D-d} \gamma^{mnp} f^{mnp} . \quad (12)$$

Simple algebra and use of the commutation relations yield

$$\mathcal{K}^2 = \left[ C_2(\mathbf{g}) + \frac{D}{24} C_2^{\text{adjt}}(\mathbf{g}) \right] - \left[ C_2(\mathbf{h}) + \frac{d}{24} C_2^{\text{adjt}}(\mathbf{h}) \right] , \quad (13)$$

which is not positive definite: the “massless” Kostant equation can have non-trivial solutions. The operators

$$S^i = -\frac{i}{4} \gamma^{mn} f^{imn} , \quad (14)$$

represent the subalgebra  $\mathbf{h}$  on the  $(D-d)$  coset space. The subalgebra  $\mathbf{h}$  is generated by the diagonal sum

$$L^i = T^i + S^i , \quad i = 1, 2, \dots, d \quad (15)$$

which commute with the Kostant operator

$$[L^i, \mathcal{K}] = 0 . \quad (16)$$

Hence, solutions of

$$\mathcal{K} \Psi = 0 , \quad (17)$$

can be assembled in full multiplets of the subalgebra  $\mathbf{h}$ .

When  $\mathbf{g}$  and  $\mathbf{h}$  have the same rank, these solutions reside in the tensor product<sup>9,11</sup>

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- , \quad (18)$$

where  $S^\pm$  are the two spinor representations of  $SO(D-d)$ , and  $V_\lambda$  is a representation of  $\mathbf{g}$  of highest weight  $\lambda$ , with all representations decomposed in terms of those of the subalgebra  $\mathbf{h}$ .

We specialize our discussion to *symmetric* coset space  $\mathbf{g}/\mathbf{h}$ , for which  $f_{mnp} = 0$ <sup>a</sup>. In that case, it is only when  $V_\lambda$  is the singlet of  $\mathbf{g}$  that there are no cancellations of representations in eq.(18). It follows that the solutions of the Kostant-Dirac equation span the two spinor representations of  $SO(D-d)$ . Since these are well-known to be generated by a Clifford algebra, they necessarily exhibit supersymmetry in terms of the subalgebra  $\mathbf{h}$ .

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<sup>a</sup>When the coset space is not symmetric,  $f_{mnp} \neq 0$ , there are always cancellations, even when  $V_\lambda$  is a singlet, and no manifest supersymmetry.

On the other hand, if  $V_\lambda$  is taken to be any non-trivial representation of  $\mathfrak{g}$ , cancellations always occur, and the solutions no longer display manifest supersymmetry.

It is also known that the number of irreps of  $\mathfrak{h}$  appearing in this difference is always the same, and equal to  $r$ , the ratio of the orders of the Weyl groups of  $\mathfrak{g}$  and  $\mathfrak{h}$ , which is also the Euler number of the coset manifold. In some cases, the r-plets, or Euler multiplets, have the same number of bosons and fermions<sup>10</sup>, but no apparent supersymmetry, which make them very interesting from a quantum mechanical point of view. We will not pursue this line of thinking in this paper. Indeed, it has been shown that

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_c^r (-1)^w U_{c \bullet \lambda} , \quad (19)$$

where the sum is over the  $r$  elements of the Weyl group of  $\mathfrak{g}$  that are not in  $\mathfrak{h}$ 's, and  $U_{c \bullet \lambda}$  denote representations of  $\mathfrak{h}$ , and the  $\bullet$  denotes a specific construction which has been defined elsewhere<sup>9,11</sup>.

To summarize, the solutions of the massless Kostant operator appear in Euler multiplets, one for each representation of  $\mathfrak{g}$ . Each Euler multiplet contains exactly  $r$  irreps of the subalgebra  $\mathfrak{h}$ . For symmetric coset spaces, the Euler multiplet associated with the singlet of  $\mathfrak{g}$  spans a Clifford algebra, and therefore displays supersymmetry.

### 3.3 Supersymmetric Euler Multiplets

It is well-known that the degrees of freedom of supersymmetric theories can be labelled in terms of the Wigner little group of the associated Poincaré algebra. Since theories with gravity cannot sustain a finite number of *massless* degrees of freedom with spin higher than two, the cosets cannot exceed sixteen dimensions, which yield the two spinor representations of  $SO(16)$ , with 256 degrees of freedom.

We have found only a few cosets, with  $D - d = 16, 8, 4$ , for which the basic Euler multiplets have the right quantum numbers to represent massless particles of relativistic theories:

#### ▷ 16-dimensional Cosets

The 256 states of the associated Clifford are generated by the two spinor irreps of  $SO(16)$ , yielding four possibilities:

- $SU(9) \supset SU(8) \times U(1)$  with lowest Euler multiplet

$$\mathbf{1}_2 \oplus \mathbf{8}_{3/2} \oplus \mathbf{28}_1 \oplus \mathbf{56}_{1/2} \oplus \mathbf{70}_0 \oplus \overline{\mathbf{56}}_{-1/2} \oplus \overline{\mathbf{28}}_{-1} \oplus \overline{\mathbf{8}}_{-3/2} \oplus \mathbf{1}_{-2}$$

Interpreting the  $U(1)$  as the helicity little group in four dimensions, these can be thought of as the massless spectrum of either type IIB string theory, or of  $N = 8$  supergravity.

- $SO(10) \supset SO(8) \times SO(2)$  with lowest Euler multiplet

$$\mathbf{1}_2 \oplus \mathbf{8}_{3/2} \oplus \mathbf{28}_1 \oplus \mathbf{56}_{1/2} \oplus (\mathbf{35}_0 \oplus \mathbf{35}_0) \oplus \mathbf{56}_{-1/2} \oplus \mathbf{28}_{-1} \oplus \mathbf{8}_{-3/2} \oplus \mathbf{1}_{-2}$$

Viewing  $SO(8)$  as the little group in ten dimensions, these may represent the massless spectrum of IIB superstring in ten dimensions, but only after using  $SO(8)$  triality. Alternatively, with  $U(1)$  as the helicity, it could just be the massless particle content of the type IIB superstring theory, or of  $N = 8$  supergravity in four dimensions.

- $SU(6) \supset SO(6) \times SO(3) \times SO(2)$  with lowest Euler multiplet

$$\begin{aligned} &(\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{4}, \mathbf{2})_{3/2} \oplus (\mathbf{10}, \mathbf{1})_1 \oplus (\mathbf{6}, \mathbf{3})_1 \oplus (\mathbf{4}, \mathbf{4})_{1/2} \oplus (\mathbf{20}, \mathbf{2})_{1/2} \\ &\oplus (\mathbf{20}', \mathbf{1})_0 \oplus (\mathbf{15}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{5})_0 \\ &\oplus (\mathbf{4}, \mathbf{4})_{-1/2} \oplus (\overline{\mathbf{20}}, \mathbf{2})_{-1/2} \oplus (\overline{\mathbf{10}}, \mathbf{1})_{-1} \oplus (\mathbf{6}, \mathbf{3})_{-1} \oplus (\overline{\mathbf{4}}, \mathbf{2})_{-3/2} \oplus (\mathbf{1}, \mathbf{1})_{-2} . \end{aligned}$$

This is the massless spectrum of  $N = 8$  supergravity, or of type IIB superstring in four dimensions, or of massless theories in five and eight dimensions.

- $F_4 \supset SO(9)$  The lowest Euler multiplet

$$\mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$$

describes the massless spectrum of  $N = 1$  supergravity in 11 dimensions, the local limit of M-theory, by identifying  $SO(9)$  as the massless little group.

#### ▷ 8-dimensional Cosets

The Clifford algebra is realized on the 16 states that span the two spinor irreps of  $SO(8)$ , leading to the two massless interpretations

- $SU(5) \supset SU(4) \times U(1)$  which yields the lowest Euler multiplet

$$\mathbf{1}_1 \oplus \mathbf{4}_{1/2} \oplus \mathbf{6}_0 \oplus \overline{\mathbf{4}}_{-1/2} \oplus \mathbf{1}_{-1}$$

With  $U(1)$  as the helicity little group, this particle content is the same as the massless spectrum of  $N = 4$  Yang-Mills in four dimensions.

It could also describe a massless theory in eight dimensions with one conjugate spinor, one vector, and two scalars, since  $SU(4) \sim SO(6)$ .



- $SO(6) \supset SO(4) \times SO(2)$  with lowest Euler multiplet

$$(\mathbf{1}, \mathbf{1})_1 \oplus (\mathbf{2}, \mathbf{2})_{1/2} \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{3}, \mathbf{1})_0 \oplus (\mathbf{2}, \mathbf{2})_{-1/2} \oplus (\mathbf{1}, \mathbf{1})_{-1}$$

This particle content is the same as the  $N = 4$  Yang-Mills in four dimensions, or the  $N = 2$  vector supermultiplet in five dimensions.

#### ▷ 4-dimensional Cosets

The Clifford algebra is realized on 4 states of the two spinor irreps of  $SO(4)$ . They appear as the lowest Euler multiplet in the following decomposition:

- $SU(3) \supset SU(2) \times U(1)$ . The lowest Euler multiplet

$$\mathbf{1}_{1/2} \oplus \mathbf{2}_0 \oplus \mathbf{1}_{-1/2}$$

can be interpreted as a relativistic theory in 4 dimensions, since the  $U(1)$  charges are half integers and integers, and it can describe the massless  $N = 1$  Wess-Zumino supermultiplet in four dimensions, as well as a massless supermultiplet in five dimensions.

So far, we have limited our discussion to massless degrees of freedom, but we should note for completeness that possible relativistic theories with massive degrees of freedom appear in the cosets  $Sp(2P + 2) \supset Sp(2P) \times Sp(2)$  with the same content as massive  $N = P$  supersymmetry in four dimensions. The massive  $N = 4$  supermultiplet in four dimensions is also described by the coset  $SO(8) \supset SO(4) \times SO(4)$ , and the massive  $N = 2$  supermultiplet in four dimensions can be described in terms of  $G_2/SU(2) \times SU(2)$ .

In this section, we have shown that the degrees of freedom of some supersymmetric theories are solutions of the Kostant-Dirac equation associated with specific cosets. While non-relativistic, this identification is suggestive of an alternate description, in which the states appear as solutions of an operator equation, not as fundamental fields in a Lagrangian. This is of course reminiscent of what happens in string theory. Furthermore, not all supersymmetric theories appear in this list, only the local limit of M-theory, of type IIB superstring theories (not type IIA, type I, nor heterotic superstrings), and certain local field theories,  $N = 4$  Yang-Mills, and  $N = 1$  Wess-Zumino multiplets. All the massless cosets are both hermitian and symmetric, except for the Wess-Zumino multiplet and  $N = 1$  supergravity in eleven dimensions which are only symmetric.

To take advantage of these alternate descriptions, the Kostant-Dirac operator must be given a fundamental role. In the next section, we propose one

approach where the Kostant-Dirac operator is the zero mode of the superconformal generator associated with the coset, constructed some years ago by Kazama and Suzuki<sup>13</sup>.

#### 4 Coset Construction of Superconformal Algebras

One can construct an affine generalization of the Kostant-Dirac operator, much in the same way the Dirac equation was generalized many years ago: one generalizes the Clifford matrices to the world sheet as

$$\gamma^a \rightarrow \Gamma^a(\sigma) , \quad (20)$$

with

$$\{\Gamma^a(\sigma), \Gamma^b(\sigma')\} = 2\delta^{ab}\delta(\sigma - \sigma') , \quad (21)$$

as well as the generators

$$T^a \rightarrow T^a(\sigma) , \quad (22)$$

which satisfy the level  $k$  Kac-Moody commutation relations

$$[T^a(\sigma), T^b(\sigma')] = if^{abi}T^i(\sigma)\delta(\sigma - \sigma') + k\delta^{ab}\delta'(\sigma - \sigma') . \quad (23)$$

These are the cosets of the full Kac-Moody algebra of affine  $\hat{\mathfrak{g}}$ , with the additional commutation relations (we limit our discussion to symmetric cosets)

$$[T^i(\sigma), T^b(\sigma')] = if^{iab}T^b(\sigma)\delta(\sigma - \sigma') , \quad (24)$$

$$[T^i(\sigma), T^j(\sigma')] = if^{ijk}T^k(\sigma)\delta(\sigma - \sigma') + k\delta^{ij}\delta'(\sigma - \sigma') . \quad (25)$$

These two algebras are independent and commute with one another

$$[\Gamma^a(\sigma), T^b(\sigma')] = 0 . \quad (26)$$

The generators

$$F(\sigma) \equiv \Gamma^a(\sigma)T^a(\sigma) = \sum_{n=-\infty}^{+\infty} F_n e^{-in\sigma} , \quad (27)$$

generate a superconformal algebra

$$\{F_n, F_m\} = 2L_{n+m} + \frac{c}{3}(n^2 - \frac{1}{4})\delta_{m+n,0} , \quad (28)$$

where the  $L_n$  generate the Virasoro algebra

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} , \quad (29)$$

and

$$[L_n, F_m] = \left(\frac{n}{2} - m\right) F_{m+n} . \quad (30)$$

Alternatively, the Virasoro generators can be summarized in

$$L(\sigma) = T^a(\sigma)T^a(\sigma) + if^{abj}\Gamma^a(\sigma)\Gamma^b(\sigma)T^j(\sigma) - k\Gamma^a(\sigma)\frac{d\Gamma^a(\sigma)}{d\sigma} , \quad (31)$$

after getting rid of a surface term. This can be rewritten as

$$L(\sigma) = L^{\mathbf{g}}(\sigma) - L^{\mathbf{h}}(\sigma) - \frac{1}{4}f^{abj}f^{cdj}\Gamma^a(\sigma)\Gamma^b(\sigma)\Gamma^c(\sigma)\Gamma^d(\sigma) , \quad (32)$$

in which

$$L^{\mathbf{g}}(\sigma) = T^A(\sigma)T^A(\sigma) - k\Gamma^A(\sigma)\frac{d\Gamma^A(\sigma)}{d\sigma} , \quad (33)$$

$$L^{\mathbf{h}}(\sigma) = \hat{T}^j(\sigma)\hat{T}^j(\sigma) - k\Gamma^i(\sigma)\frac{d\Gamma^i(\sigma)}{d\sigma} , \quad (34)$$

where

$$\hat{T}^j(\sigma) = T^j(\sigma) - \frac{i}{2}f^{abj}\Gamma^a(\sigma)\Gamma^b(\sigma) , \quad (35)$$

generate the diagonal Kac-Moody associated with the subalgebra  $\mathbf{h}$ . This is the construction of Kazama and Suzuki<sup>13</sup>, for which we have

$$c = \frac{3}{2}\dim(\mathbf{g}/\mathbf{h})\frac{k}{k+g} , \quad (36)$$

where  $g$  is the second Dynkin index of the adjoint representation of  $\mathbf{g}$  (dual Coxeter number), and  $k$  is the level of the  $\hat{\mathbf{g}}$  Kac-Moody algebra.

It is convenient to introduce the expansions

$$\Gamma^a(\sigma) = \gamma^a + \sqrt{2}\gamma_5 \sum_{n \neq 0} b_n^a e^{-in\sigma} , \quad (37)$$

where

$$\{\gamma^a, \gamma_5\} = 0 , \quad \{b_n^a, b_m^b\} = \delta^{ab}\delta_{m+n,0} , \quad (38)$$

and

$$T^a(\sigma) = T^a + \sum_{n \neq 0} T_n^a e^{-in\sigma} , \quad (39)$$

with

$$[T_n^a, T_m^b] = if^{abj}T_{m+n}^j + km\delta^{ab}\delta_{m+n,0} . \quad (40)$$

We then see that

$$F_0 = \mathcal{K} + \sqrt{2}\gamma_5 \sum_{n \neq 0} b_n^a T_{-n}^a, \quad (41)$$

is the natural generalization of the Kostant operator. The oscillator ground states satisfy

$$b_n^a |\Psi_0\rangle = 0, \quad T_n^a |\Psi_0\rangle = 0; \quad n > 0. \quad (42)$$

Hence, the massless degrees of freedom of the previous section, are the ground states of the superconformal algebra in the fermion sector. These states form Euler multiplets of representations of  $\mathfrak{h}$ , but they contain no oscillator modes.

The oscillator modes of this superconformal algebra come from two places, the  $b_n^a$  which appear in the generalized Dirac matrices, and transform as spinors of  $\mathfrak{h}$ , and the modes which appear in the construction of the  $\hat{\mathfrak{g}}$  Kac-Moody algebra. There are several ways to generate this algebra<sup>12</sup>, by means of:

- a current algebra as a bilinear in fermion oscillators transforming as some representation of  $\mathfrak{g}$ , in which case,  $k$  is the second-order Dynkin index of the representation.
- a bosonic WZNW construction.
- a vertex construction, only if  $\mathfrak{g}$  is simply-laced, with  $k = 1$ .

Each of these constructions yields a definite level  $k$ , and therefore a different  $c$ -number for the associated superconformal algebra.

The role of the  $c$ -number in this work is not yet clear, since the use of the superconformal algebra advocated here is hardly conventional: the unbroken subgroup  $\mathfrak{h}$ , or parts thereof is not an internal symmetry group, but a *space* group. As a consequence *both space-time bosons and fermions reside in the R-sector*. On the other hand, the description is non-relativistic: the Hilbert space is manifestly positive definite, but relativistic invariance is highly non-trivial to prove, and we surmise that relativistic invariance could be possible, if at all, only for special values of the anomaly.

In this work, we do not attempt a proof of relativistic invariance, but only offer two examples which set forth *necessary* conditions for these theories to be relativistic.

Consider the lowest Euler multiplet of the  $F_4/SO(9)$  coset theory, which describes the massless degrees of freedom of the  $N = 1$  Supergravity in eleven dimensions, with  $SO(9)$  identified as the massless little group. The oscillators of this theory come from two places, those in the generalized Dirac matrices,

which transforms as the real **16** spinor of  $SO(9)$ , and the oscillators which build up the  $F_4$  Kac-Moody algebra.

If the oscillator modes of this superconformal theory are to describe massive degrees of freedom, a necessary condition for a relativistic description is that they assemble themselves in representations of the *massive* little group,  $SO(10)$ . As we see in the two examples below, this depends on the realization of the Kac-Moody algebra:

- First, we assume a current algebra realization of the  $F_4$  Kac-Moody algebra, with the fermions transforming as the lowest dimensional representation, the real **26**. In terms of  $SO(9)$ ,  $\mathbf{26} = \mathbf{1} \oplus \mathbf{9} \oplus \mathbf{16}$ . Thus the Hilbert space of this superconformal coset algebra is generated by
  - two oscillators, each transforming as the real **16** of  $SO(9)$ ,
  - one  $SO(9)$  vector oscillator
  - one  $SO(9)$  singlet oscillator.

These degrees of freedom could be assembled in representations of  $SO(10)$ : the two real spinors into the *complex* **16** spinor of  $SO(10)$ , and the singlet and  $SO(9)$  vector into the vector **10** of  $SO(10)$ . This counting does not prove that this theory can be made relativistic, only that some necessary conditions are met.

- As a second example, consider a current algebra with fermions in the **52** of  $F_4$ . Since  $\mathbf{52} = \mathbf{36} \oplus \mathbf{16}$ , we end up with
  - two oscillators, each transforming as the real **16** of  $SO(9)$ ,
  - one oscillator transforming as the **36**.

To obtain a representation of  $SO(10)$ , we would need an  $SO(9)$  vector in to complete the **45** of  $SO(10)$ . Thus a relativistic description seems implausible in this case.

Those two cases are distinguished by their Kac-Moody level, and thus by the  $c$ -number of the consequent Virasoro algebra. One value may lead to a relativistic theory, while another cannot.

Finally, we note the values of the conformal anomaly for the cosets we have listed, suggesting possible connections between (massless and massive) theories:

- $c = 24k/(k+9)$  for  $SU(9)/SU(8) \times U(1)$  and  $F_4/SO(9)$ .
- $c = 24k/(k+8)$  for  $SO(10)/SO(8) \times U(1)$ .

- $c = 24k/(k+6)$  for  $Sp(10)/Sp(8) \times Sp(2)$ .
- $c = 12k/(k+5)$  for  $SU(5)/SU(4) \times U(1)$ .
- $c = 12k/(k+4)$  for  $SO(6)/SO(4) \times SO(2)$ ,  $Sp(6)/Sp(4) \times Sp(2)$ , and  $G_2/SU(2) \times SU(2)$ .
- $c = 6k/(k+3)$  for  $SU(3)/SU(2) \times U(1)$ .

Work is in progress to determine the representations of these superconformal algebras, in order to see if their spectrum displays relativistic properties.

## 5 Euler Multiplets for $SU(3)/SU(2) \times U(1)$

We know from the work of Lerche et al <sup>11</sup>, Kostant <sup>8</sup> and GKRS <sup>9</sup> that for this equal rank embedding, there are an infinite number of Euler triples. Associated with the  $[a_1 \ a_2]$  representation of  $SU(3)$  (in Dynkinese), the solutions consist of the  $SU(2) \times U(1)$  multiplets

$$[a_1 + a_2 + 1]_{(a_1 - a_2)/6} \oplus [a_1]_{(2a_2 + a_1 + 3)/6} \oplus [a_2]_{-(2a_1 + a_2 + 3)/6} . \quad (43)$$

It is instructive to show in detail how these solutions arise.

It is convenient to act on the four-dimensional coset with the  $4 \times 4$  matrices

$$\gamma_4 = \sigma_1 \times \sigma_1 ; \quad \gamma_5 = \sigma_1 \times \sigma_2 ; \quad \gamma_6 = \sigma_1 \times \sigma_3 ; \quad \gamma_7 = \sigma_2 \times 1 , \quad (44)$$

which satisfy the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} , \quad (45)$$

for  $a, b = 4, 5, 6, 7$ . The representation of the  $SU(2) \times U(1)$  subgroup on the spinors is given by

$$S^j = -\frac{i}{4}\gamma^{ab}f^{abj} , \quad S^8 = -\frac{i}{4}\gamma^{ab}f^{8ab} , \quad (46)$$

where  $j = 1, 2, 3$ , and  $f^{jab}$  and  $f^{8ab}$  are the antisymmetric  $SU(3)$  structure functions which appear in

$$[T^A, T^B] = if^{ABC}T^C , \quad (47)$$

with  $A, B, C = 1, 2, \dots, 8$ , with  $f^{123} = 1$ , and other nonzero values

$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} ; \quad f^{845} = f^{867} = \frac{\sqrt{3}}{2} . \quad (48)$$

The Kostant operator is given by

$$\mathcal{K} \equiv \sum_{a=4}^7 \gamma^a T^a , \quad (49)$$

and is invariant under the action of the diagonal generators of the  $SU(2) \times U(1)$

$$L^j = T^j + S^j , \quad L^8 = T^8 + S^8 . \quad (50)$$

To solve the Kostant-Dirac equation

$$\mathcal{K} \Psi = 0 , \quad (51)$$

introduce

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \quad (52)$$

where  $\psi_{1,2}$  are two-component spinors which satisfy

$$\begin{aligned} (\sigma_1 T^4 + \sigma_2 T^5 + \sigma_3 T^6 + iT^7) \psi_1 &= 0 , \\ (\sigma_1 T^4 + \sigma_2 T^5 + \sigma_3 T^6 - iT^7) \psi_2 &= 0 . \end{aligned} \quad (53)$$

These equations are most easily worked out in the Schwinger representation of the  $SU(3)$  generators

$$T^A = \mathbf{a}^\dagger \frac{\lambda^A}{2} \mathbf{a} , \quad (54)$$

where the  $\lambda^A$  are the Gell-Mann matrices, and  $\mathbf{a}(\mathbf{a}^\dagger)$  represent three ladder operators  $a_\alpha(a_\alpha^\dagger)$ , with the standard commutation relations

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta} . \quad (55)$$

If we further split the components as

$$\psi_{1,2} = \begin{pmatrix} \eta_{1,2}^+ \\ \eta_{1,2}^- \end{pmatrix} , \quad (56)$$

these equations become

$$\begin{pmatrix} a_3^\dagger a_2 & a_3^\dagger a_1 \\ a_1^\dagger a_3 & -a_2^\dagger a_3 \end{pmatrix} \begin{pmatrix} \eta_2^+ \\ \eta_2^- \end{pmatrix} = 0 , \quad \begin{pmatrix} a_2^\dagger a_3 & a_3^\dagger a_1 \\ a_1^\dagger a_3 & -a_3^\dagger a_2 \end{pmatrix} \begin{pmatrix} \eta_1^+ \\ \eta_1^- \end{pmatrix} = 0 , \quad (57)$$

For each representation of  $SU(3)$ , there are three solutions of the Kostant-Dirac equations. The simplest solutions are just constant entries times the oscillator vacuum state. Under the diagonal subgroup, they split into two singlets

$$|\xi\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} |0\rangle, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} |0\rangle, \quad (58)$$

and one doublet

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} |0\rangle, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} |0\rangle. \quad (59)$$

The second Euler triplet solutions are made up of states with one oscillator. They consist of one triplet with  $L_8 = 1/(2\sqrt{3})$

$$\begin{pmatrix} 0 \\ 0 \\ a_1^\dagger \\ 0 \end{pmatrix} |0\rangle, \quad \begin{pmatrix} 0 \\ 0 \\ a_2^\dagger \\ -a_1^\dagger \end{pmatrix} |0\rangle, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_1^\dagger \end{pmatrix} |0\rangle, \quad (60)$$

one doublet with  $L_8 = 2/\sqrt{3}$  and one singlet with  $L_8 = -5/(2\sqrt{3})$

$$\begin{pmatrix} a_1^\dagger \\ 0 \\ 0 \\ 0 \end{pmatrix} |0\rangle, \quad \begin{pmatrix} a_2^\dagger \\ 0 \\ 0 \\ 0 \end{pmatrix} |0\rangle; \quad \begin{pmatrix} 0 \\ a_3^\dagger \\ 0 \\ 0 \end{pmatrix} |0\rangle. \quad (61)$$

The ground state Euler multiplet can be generated by two nilpotent operators  $Q_\alpha$ , with  $T_8 = -\sqrt{3}/2$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (62)$$

They anticommute

$$\{Q_\alpha, Q_\beta\} = 0, \quad (63)$$

and satisfy

$$[T_i, Q_\alpha] = \frac{i}{2}(\sigma_i)_\alpha^\beta Q_\beta, \quad [T_8, Q_\alpha] = -i\frac{\sqrt{3}}{2}Q_\alpha, \quad (64)$$



so that the ground states can be written in the usual supersymmetric notation as

$$|\xi\rangle, \quad Q_1|\xi\rangle, \quad Q_2|\xi\rangle, \quad Q_1Q_2|\xi\rangle. \quad (65)$$

It does not seem possible to write the states at the next level in the same way; yet they carry much of the same structure as those of the ground state.

One can build ladder operators which make transitions between one triple and the next. There are two such operators, a doublet  $D_\alpha$  with  $T_8 = 1/(2\sqrt{3})$ , and a singlet  $S$  with  $T_8 = -1/\sqrt{3}$ , with matrix representation

$$D_{1,2} = \begin{pmatrix} a_{1,2}^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1,2}^\dagger & 0 \\ 0 & 0 & 0 & a_{1,2}^\dagger \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_3^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (66)$$

Let us normalize the value of  $T_8$  so as to give the singlets in the lowest triple values of  $\pm 1/2$  which are natural if one wants to think of the  $U(1)$  as the Poincaré group helicity. Then the ladder operators have “helicities” of  $1/6$  and  $-1/3$ , respectively, which makes little sense, unless we consider the continuous spin representations of the Poincaré group. It is therefore doubtful that these operators can be explained in the usual relativistic framework.

## 6 Conclusions

In this paper, we have drawn attention to certain intriguing facts:

- The massless degrees of freedom of certain local theories, some of which are local limits of more elaborate theories (M-theory, type IIB Superstring theory), can be viewed as solutions of an algebraic Dirac-like equation. This raises the possibility of an alternate formulation for these theories.
- This Dirac-like equation can be identified with the zero-mode of the superconformal generator in the Kazama-Suzuki coset construction. These might therefore stem from theories with an underlying two-dimensional structure.

Many questions remain unanswered, notably,

- How is the ground state Euler multiplet selected?
- What is the role of the other Euler multiplets?

- What is the spectrum of these superconformal theories?
- Is their spectrum relativistic, and under what conditions?
- What are the interactions?

Until these questions are answered, one cannot claim that the suggestions in this paper are physically relevant, but our list includes many theories of great interest, namely  $N = 4$  super-Yang-Mills, type IIB superstring theory, and most notably,  $N = 1$  supergravity in eleven dimensions. The latter is the local limit of M-theory, which raises the possibility that M-theory also displays an underlying two-dimensional structure, (perhaps holographically related to a membrane?). Equally intriguing is the appearance of the exceptional group  $F_4$  in the description of its *space* degrees of freedom of this theory.

*We dedicate this work to the memory of the late Professor Golfand.*

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